Phase transitions in periodically driven macroscopic systems

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We study the large-time behavior of a class of periodically driven macroscopic systems. We find, for a certain range of the parameters of either the system or the driving fields, the time-averaged asymptotic behavior effectively is that of certain other equilibrium systems. We then illustrate with a few examples how the conventional knowledge of the equilibrium systems can be made use of in choosing the driving fields to engineer new phases and to induce new phase transitions.

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I. INTRODUCTION

With enduring interest, studies have been pursued to understand systems with many degrees of freedom at, near, and far away from equilibrium. Systems at equilibrium are studied within the well-established Boltzmann-Gibbs framework, while those near and far from equilibrium lack such a framework and are usually described by stochastic dynamical equations. The stochastic equations that describe the dynamics near equilibrium need to ensure that the system relaxes to equilibrium and hence are less difficult to construct than when the system is far away from equilibrium. However, this condition that the asymptotic solution is the equilibrium distribution does not provide a unique stochastic equation. In fact, for a given equilibrium system infinitely many such equations near equilibrium can be provided that would lead to the same static properties [1]. This naturally motivates the study of various stochastic models.

We know that a closed system, characterized by the Hamiltonian H, when exposed to the environment for a long time, will equilibriate and be described by the Boltzmann distribution $\exp(-\beta H)$. If this system is also subjected to an external periodic force then what is its large-time behavior? The absence of an established framework to find this asymptotic behavior and the fact that the equilibrium behavior is describable by the asymptotic solution of certain stochastic equations, makes it essential to study various periodically driven stochastic systems. It is also essential as some of these systems show interesting behavior and some of them furnish useful applications. Stochastic resonance [2] and magnetic hysteresis [3] are some of the well known phenomena that are exhibited by certain periodically driven stochastic systems.

A large number of stochastic processes have been studied for many decades now [4]. Studies in critical dynamics have been actively pursued [5] after renormalization group techniques were successfully applied to equilibrium systems [6]. Many driven diffusive models were studied [7] and the critical behavior of some of them were analyzed too [8,9]. Periodically driven stochastic particle systems were extensively studied in the context of stochastic resonance [2], while similar studies on fields were relatively few.

The issues that we address in this paper are related to a class of periodically driven macroscopic systems that, in the absence of driving, would relax to equilibrium. The relevant degrees are described by a field and the equilibrium properties are characterized by an energy functional of the field. These systems may or may not have a critical point when in equilibrium but could exhibit a critical behavior when the driving fields are switched on. How do these macroscopic systems respond to periodically driven fields? How do we describe the phases of the driven system? What is the dependence of the critical points, if any, on the parameters of the driving fields? Can driving change the nature of the phase transitions? Can it lead to new phases?

The layout of the paper is as follows. In the next section, we define a class of stochastic models to describe the relevant fields of periodically driven systems. In Sec. III, we develop a perturbative scheme to solve the Fokker-Planck equation that describes the systems of our interest and explicitly find their asymptotic behavior to first order. In Sec. IV, we illustrate with examples some of the effects of driving on phases and phase transitions.

II. STOCHASTIC MODELS FOR PERIODICALLY DRIVEN SYSTEMS

In this section, we will define a class of periodically driven stochastic systems. We will assume the slowly relaxing modes of the system in the absence of the periodic forces to be the relevant degrees of freedom and then model the dynamics of these variables by modifying the stochastic equations of these modes in the presence of the periodic forces.

Let us first recollect how the dynamics near equilibrium is described in the absence of driving. In this case the variables of interest, those that relax very slowly, are the order parameter and the conserved quantities. This is a small set of variables which are some functions of the original degrees of freedom. In principle, the dynamics of these relevant variables can be obtained from the equations of motion of the original degrees, by integrating out the unwanted variables. In practice, these dynamical equations for the relevant degrees are not thus established as they are not usually derivable due to the large number of unwanted variables involved. Hence these equations are based on certain guiding principles [10]. (i) Foregoing a large number of unwanted variables renders a stochastic evolution to the relevant ones. (ii)

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These stochastic dynamical equations should be such that the large-time distribution gives the right static thermodynamic properties. This large-time distribution is the distribution either obtained by integrating out the irrelevant variables from the canonical equilibrium distribution or found on phenomenological grounds, e.g., Landau-Ginzburg (LG) theory near critical point. Finally, the stochastic equations thus established could explain the dynamic properties of the relevant variables near equilibrium.

There exist many different families of stochastic equations that give the same static properties for the relevant degrees. One of these is the following Langevin equation, the time-dependent Landau-Ginzburg (TDLG) theory that describes the dynamics near equilibrium

$$\Gamma\frac{\partial}{\partial t}\varphi(x,t) = -\left.\frac{\delta H(\varphi)}{\delta\varphi(x)}\right|_{\varphi(x)\to\varphi(x,t)} + \eta(x,t), \qquad (1)$$

where $\varphi(x,t)$ is a $(d+1)$-dimensional stochastic field that relaxes to the order parameter $\varphi(x)$ of the system and $\mathcal{H}(\varphi)$ is the LG free-energy functional; $\eta(x,t)$ is a Gaussian random field with $\langle\eta(x,t)\rangle_\eta=0$ and $\langle\eta(x,t)\eta(x',t')\rangle_\eta=2\Gamma\delta(x-x')\delta(t-t')$. If $\varphi_\eta(x,t)$ is the solution of the above Langevin equation then the probability distribution of this solution $P(\varphi,t)=\Pi_x\langle\delta(\varphi(x)-\varphi_\eta(x,t))\rangle_\eta$, evolves according to the Fokker-Planck (FP) equation $\partial_t P=\mathcal{L}P$ that is obtained from Eq. (1) and the Gaussian distribution of $\eta(x,t)$. The formal solution of this FP equation is $P(\varphi,t)=\exp(\mathcal{L}t)P(\varphi,t=0)$, where $P(\varphi,t=0)$ is the initial distribution. The asymptotic distribution $P_\infty(\varphi,t)=\lim_{t\to\infty}P(\varphi,t)$ is the right eigenfunction of the $\mathcal{L}$ operator with zero eigenvalue. This asymptotic probability distribution is the LG measure $P_{\mathrm{eq}}(\varphi)\sim\exp(-\mathcal{H}(\varphi))$, provided (i) the eigenvalues of $\mathcal{L}$ are negative semidefinite, (ii) there exists a unique normalizable eigenfunction corresponding to zero eigenvalue in the connected space of functions to which the initial distribution belongs to, and (iii) the initial distribution is not orthogonal to this eigenfunction.

The presence of driving fields on the system will alter both its kinematics and dynamics. Assumptions about kinematics follow. (i) The set of variables that were relevant in the absence of driving for the dynamics near equilibrium continue to remain relevant upon introducing driving. (ii) Some minimal set of additional variables might become relevant too. We will consider a velocity field $\pi(x)$ along with the order parameter field $\varphi(x)$ to be relevant. The dynamical variables of the system are then specified by $\varphi(x,t)$ and its time derivative $\partial_t\varphi(x,t)$. Assumptions about the dynamics are as follows. (i) The Langevin equation, that describes dynamics near equilibrium in the absence of driving, gets modified by adding terms related to the periodically driven fields and by adding a second-order time-derivate term $\partial_t^2\varphi(x,t)$ whose effect is significant when the driving frequency is high. (ii) The periodic driving does not change the properties of the noise field for any frequency of the driving fields.

The modified TDLG theory that describes the system when subjected to driving is then given by the following stochastic equation:

$$m\frac{\partial^2}{\partial t^2}\varphi(x,t) = -\Gamma\frac{\partial}{\partial t}\varphi(x,t) - \frac{\delta\mathcal{H}(\varphi)}{\delta\varphi(x,t)} + \mathcal{F}(\varphi(x,t),t) + \eta(x,t), \qquad (2)$$

where $\mathcal{F}(\varphi(x),t)$ is the driving field that is periodic in time with period $T=2\pi/\Omega$. The general form of this field is

$$\mathcal{F}(\varphi(x),t) = \sum_{n=1}^{\infty}\left[F_n(\varphi(x))\cos(n\Omega t) + G_n(\varphi(x))\sin(n\Omega t)\right], \qquad (3)$$

where F_n and G_n are arbitrary local functionals of $\varphi(x)$. This term could be thought of as a result of making the coupling constants in $\mathcal{H}(\varphi)$ time dependent and periodic. We will take $m=1$ so that the expressions look simple and to regain the m dependence back replace $t\to t/\sqrt{m}$, $\Gamma\to\Gamma/\sqrt{m}$, and $\Omega\to\sqrt{m}\Omega$. The distribution of the noise field is Gaussian as specified earlier.

The above dynamical equation for a $(0+1)$-dimensional stochastic field describes periodically driven Brownian particle. These Brownian particles exhibit a variety of interesting asymptotic behavior depending on the driving forces [11]: The particles when driven could congregate around more than one point even though when in equilibrium they would have congregated around a single point. Particles with different masses respond to driving differently and thus when they are mixed and driven would cluster around different points.

The phase space probability distribution is defined as $P(\varphi,\pi,t)=\langle\Pi_x\delta(\varphi(x)-\varphi_\eta(x,t))\delta(\pi(x)-\partial_t\varphi_\eta(x,t))\rangle_\eta$, where $\varphi_\eta(x,t)$ is the solution of Eq. (2) at time t for a particular history of $\{\eta(x,t)\}$ over a time t and $\langle\cdots\rangle_\eta$ is the average over the noise distribution. This distribution satisfies the normalization condition $\int\mathcal{D}[\varphi,\pi]P(\varphi,\pi,t)=1$. The time evolution of $P(\varphi,\pi,t)$ is described by the FP equation

$$\frac{\partial}{\partial t}P(\varphi,\pi,t) = \mathcal{L}(t)P(\varphi,\pi,t), \qquad (4)$$

where

$$\mathcal{L}(t) = \int_x\left[-\frac{\delta}{\delta\varphi}\pi - \frac{\delta}{\delta\pi}\left(-\Gamma\pi - \frac{\delta\mathcal{H}}{\delta\varphi} + \mathcal{F}(\varphi,t)\right) + \Gamma\frac{\delta^2}{\delta\pi^2}\right]$$
$$= \mathcal{L}_{\mathrm{FP}} - \int_x\frac{\delta}{\delta\pi}\mathcal{F}(\varphi,t). \qquad (5)$$

The FP operator $\mathcal{L}_{\mathrm{FP}}$ is defined as the time-independent part of the $\mathcal{L}(t)$ operator, both of which are assumed to be suitably regularized. To keep the notation compact, $\varphi=\varphi(x)$ and $\pi=\pi(x)$ are used in the above equation. The solution of this equation is not known in general and our aim is to find the asymptotic solution for some range of parameters of the system and driving fields.

The asymptotic distribution of the above FP equation is a periodic function with period T, if the real part of the eigenvalues of $\partial_t - \mathcal{L}(t)$ are positive semidefinite. In brief, the argument for the periodicity goes as follows. The solution of the FP equation is the right eigenfunction of the operator $\partial_t - \mathcal{L}(t)$ corresponding to the zero eigenvalue. If $\mathcal{L}(t)=\mathcal{L}(t+T)$ then $\partial_t - \mathcal{L}(t)$ commutes with the discrete time-translation operator $\exp(T\partial_t)$. The solution can then be expanded in terms of the common right eigenfunctions of these two operators. Let the eigenfunctions of $\partial_t - \mathcal{L}(t)$ and $\exp(T\partial_t)$ with eigenvalues 0 and $\exp(\mu T)$, respectively, be the Floquet-type functions $\exp(\mu t)p_\mu(t)$, where $p_\mu(t)$ is a periodic function with period T. Substituting these eigenfunctions in FP equation gives $[\partial_t - \mathcal{L}(t)]p_\mu(t) = -\mu p_\mu(t)$. Hence, if the real part of the eigenvalues of $\partial_t - \mathcal{L}(t)$ are positive definite then in the limit $t\to\infty$ the only eigenfunction that survives is $p_0(t)$, thus making the asymptotic distribution periodic.

When the coupling constants $\{\tilde{g}\}$ of the driving field $\mathcal{F}(\varphi,t)$ are small compared to the coupling constants $\{g\}$ of $\mathcal{H}(\varphi)$ then the FP equation can be solved perturbative in $\{\tilde{g}\}$ provided we know the right and left eigenfunctions of the FP operator $\mathcal{L}_{\rm FP}$. These eigenfunctions are generically not known though the eigenfunctions of the FP operator which includes only the free part of $\mathcal{H}(\varphi)$ are obtainable. Hence, the eigenfunctions of $\mathcal{L}_{\rm FP}$ can be determined perturbatively in $\{g\}$ and in turn the solution of the FP equation in the double series expansion in $\{g\}$ and $\{\tilde{g}\}$. We will now further to find the asymptotic behavior of the FP equation when $\{\tilde{g}\}$ are not necessarily small compared to $\{g\}$.

III. ASYMPTOTIC DISTRIBUTION OF THE FP EQUATION

In this section, we will first transform the FP equation to enable us to have a non-perturbative solution in $\{\tilde{g}\}$ but perturbative in a parameter that involves both $\{\tilde{g}\}$ and $\Omega^2+\Gamma^2$. We then explicitly evaluate the asymptotic distribution to first order in this parameter and more specifically the time-averaged asymptotic correlation functions. We find that these correlation functions can be expressed as equilibrium correlation functions with an effective LG energy functional.

A. Formal solution

We will make a change of variables and transform the FP equation such that the time-dependent part of the transformed FP operator is small though the time-dependent part of the original FP equation is not. The variables $\{\varphi,\pi,t\}$ are changed to $\{\Phi,\Pi,\tau\}$ by the following nonlinear transformation:

$$\varphi(x) = \Phi(x) + \xi(x;\Phi,\tau),$$

$$\pi(x) = \Pi(x) + \partial_\tau \xi(x;\Phi,\tau),$$

$$t = \tau, \tag{6}$$

where the explicit form of $\xi(x;\Phi,\tau)$ will be specified later. Under this transformation the probability distribution is made to behave as a scalar: $P(\varphi,\pi,t)\to\widetilde{P}(\Phi,\Pi,\tau) = P(\varphi,\pi,t)$. The functional derivatives will then transform as

$$\frac{\delta}{\delta\pi(x)} = \frac{\delta}{\delta\Pi(x)},$$

$$\frac{\delta}{\delta\varphi(x)} = \int_y D(x,y)\left(\frac{\delta}{\delta\Phi(y)} - \int_z \partial_\tau M(y,z)\frac{\delta}{\delta\Pi(z)}\right), \tag{7}$$

and the time derivative as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial\tau} - \int_x \partial_\tau^2 \xi(x;\Phi,\tau)\frac{\delta}{\delta\Pi(x)} - \int_x \partial_\tau\xi(x;\Phi,\tau)\frac{\delta}{\delta\varphi(x)}, \tag{8}$$

where

$$M(x,y) = \frac{\delta}{\delta\Phi(x)}\xi(y;\Phi,\tau),\ \int_y D(x,y)\frac{\delta\varphi(z)}{\delta\Phi(y)} = \delta(x-z). \tag{9}$$

Substituting the above new variables in Eq. (4) we obtain the following transformed FP equation:

$$\begin{aligned}\frac{\partial\widetilde{P}}{\partial\tau} = \int_x \Bigg[-\frac{\delta}{\delta\Phi}\Pi - \frac{\delta}{\delta\Pi}(-\Gamma\Pi + f_{\mathcal{H}}(\Phi+\xi) + \Delta\mathcal{F}(\xi,\tau)) \\ + \Gamma\frac{\delta^2}{\delta\Pi^2}\Bigg]\widetilde{P} + \int_{x,y}\Pi(x)\Bigg[\{\delta(x-y) - D(x,y)\}\frac{\delta}{\delta\Phi(y)} \\ + D(x,y)\int_z \partial_\tau M(y,z)\frac{\delta}{\delta\Pi(z)}\Bigg]\widetilde{P} + \int_x\Bigg[\big(\partial_\tau^2\xi + \Gamma\partial_\tau\xi \\ - \mathcal{F}(\Phi,\tau)\big)\frac{\delta}{\delta\Pi(x)}\Bigg]\widetilde{P},\end{aligned} \tag{10}$$

where $f_{\mathcal{H}}(\varphi(x)) = -\delta\mathcal{H}/\delta\varphi(x)$ and $\Delta\mathcal{F}(\xi,\tau) = \mathcal{F}(\Phi+\xi,\tau) - \mathcal{F}(\Phi,\tau)$. We now choose $\xi(x;\Phi,\tau)$ to be the solution of the following equation:

$$\partial_\tau^2\xi + \Gamma\partial_\tau\xi - \mathcal{F}(\Phi,\tau) = 0, \tag{11}$$

whose explicit form is

$$\begin{aligned}\xi(x;\Phi,\tau) = \sum_{n=1}^{\infty}\frac{-1}{n^2\Omega^2+\Gamma^2}\Bigg[\bigg(F_n(\Phi(x)) \\ + \frac{\Gamma}{n\Omega}G_n(\Phi(x))\bigg)\cos(n\Omega\tau) + \bigg(G_n(\Phi(x)) \\ - \frac{\Gamma}{n\Omega}F_n(\Phi(x))\bigg)\sin(n\Omega\tau)\Bigg].\end{aligned} \tag{12}$$

The reason for this specific choice for ξ is that not only does it make the last term on the right-hand side of Eq. (10) zero but also makes the (time)τ-dependent part of the modified FP operator to be of $O(\xi)$. Thus if ξ is small then this modified equation will yield a perturbative solution. Upon substituting ξ from Eq. (12) in Eq. (10) we finally get

$$\frac{\partial}{\partial\tau}\widetilde{P}(\Phi,\Pi,\tau)=[\mathcal{L}+\Delta\mathcal{L}]\widetilde{P}(\Phi,\Pi,\tau), \tag{13}$$

where $\mathcal{L}$ is the following static FP operator:

$$\mathcal{L}=\int_x\left[-\frac{\delta}{\delta\Phi}\Pi-\frac{\delta}{\delta\Pi}(-\Gamma\Pi+\overline{f_{\mathcal{H}}(\Phi+\xi)}+\overline{\Delta\mathcal{F}(\xi,\tau)})+\Gamma\frac{\delta^2}{\delta\Pi^2}\right], \tag{14}$$

and $\Delta\mathcal{L}$ is the periodic (time) τ-dependent operator

$$\Delta\mathcal{L}=\int_x\left[-\frac{\delta}{\delta\Pi}(f_{\mathcal{H}}(\Phi+\xi)-\overline{f_{\mathcal{H}}(\Phi+\xi)}+\Delta\mathcal{F}(\xi,\tau)-\overline{\Delta\mathcal{F}(\xi,\tau)})\right]+\int_{x,y}\Pi(x)\left[\{\delta(x-y)-D(x,y)\}\frac{\delta}{\delta\Phi(y)}+D(x,y)\int_z\partial_\tau M(y,z)\frac{\delta}{\delta\Pi(z)}\right], \tag{15}$$

where the overbars indicate an average over a time period.

The perturbative asymptotic solution can be formally written as

$$\widetilde{P}_\infty(\Phi,\Pi,\tau)=Q(\Phi,\Pi)+\frac{1}{\partial_\tau-\mathcal{L}}\Delta\mathcal{L}\widetilde{P}_\infty(\Phi,\Pi,\tau), \tag{16}$$

where $Q(\Phi,\Pi)$ is the right eigenfunction of $\mathcal{L}$ with zero eigenvalue. If $\mathcal{L}$ has a unique eigenfunction corresponding to zero eigenvalue, and the real part of the nonzero eigenvalues do not vanish, and $\widetilde{P}_\infty$ is periodic in time, then it follows from Eq. (13) that $\Delta\mathcal{L}\widetilde{P}_\infty$ has no overlap with the eigenfunction of $\partial_\tau-\mathcal{L}$ corresponding to zero eigenvalue. Thus, though $\partial_\tau-\mathcal{L}$ is not invertible, its inverse action on the space orthogonal to its eigenfunction with zero eigenvalue is well defined.

B. Effective theory

In this subsection, we will obtain the effective energy functional to $\mathcal{O}(\xi)$ by averaging the asymptotic correlation functions over a time period. The observables will be related to the time-averaged correlation functions if the time of observation is comparable to the time period of the driving fields. In other words, we are assuming that the driving fields oscillate rapidly compared to the time scale of the measurement.

The equal-time correlation functions of the stochastic field transform under the change of variables as

$$\langle\varphi(x_1,t)\varphi(x_2,t)\cdots\rangle_\eta=\int\mathcal{D}[\pi,\varphi]\varphi(x_1)\varphi(x_2)\cdots P(\varphi,\pi,t)=\int\mathcal{D}[\Pi,\Phi]J[\Phi]\{\Phi(x_1)+\xi(x_1;\Phi,t)\}\times\{\Phi(x_2)+\xi(x_2;\Phi,t)\}\cdots\widetilde{P}(\Phi,\Pi,t), \tag{17}$$

where the integration measure $\mathcal{D}[\pi,\varphi]=\Pi_x[d\pi(x)d\varphi(x)]$ and the Jacobian of the transformation $J[\Phi]=\Pi_x[1+\partial\xi(x,\Phi(x),t)/\partial\Phi(x)]$. The first-order asymptotic distribution, obtained from Eq. (16), is $\widetilde{P}_\infty=Q^{(1)}+(\partial_\tau-\mathcal{L})^{-1}\Delta\mathcal{L}Q^{(0)}$, where $Q^{(0)}$ and $Q^{(1)}$ are the solutions of the equation $\mathcal{L}Q=0$ to $O(1)$ and $O(\xi)$, respectively. Substituting $\widetilde{P}_\infty$ in Eq. (17) and averaging over a period gives

$$\int\mathcal{D}[\pi,\varphi]\varphi(x_1)\varphi(x_2)\cdots\overline{P_\infty[\varphi,\pi,t]}=\int\mathcal{D}[\Pi,\Phi]\Phi(x_1)\Phi(x_2)\cdots Q^{(1)}(\Phi,\Pi)+O(\xi^2), \tag{18}$$

since the time average of both ξ and $[(\partial_\tau-\mathcal{L})^{-1}\Delta\mathcal{L}Q^{(0)}]$ are zero. From Eq. (14), in which $\overline{f_{\mathcal{H}}(\Phi+\xi)}$ and $\overline{\Delta\mathcal{F}(\xi,\tau)}$ are expanded to $O(\xi)$, we get

$$Q^{(1)}(\Phi,\Pi)=\frac{1}{Z^{(1)}}\exp\left(-\frac{1}{2}\int_x\Pi^2(x)-\mathcal{H}(\Phi)-\Delta\mathcal{H}(\Phi)\right), \tag{19}$$

where $Z^{(1)}$ is the normalization constant and $\Delta\mathcal{H}(\Phi)$ satisfies the condition

$$\frac{\delta}{\delta\Phi(x)}\Delta\mathcal{H}(\Phi)=-\overline{\xi(x;\Phi,t)\frac{\partial}{\partial\Phi(x)}\mathcal{F}(\Phi(x),t)}. \tag{20}$$

Substituting ξ and $\mathcal{F}$ from Eqs. (12) and (3) in the above equation we get $\Delta\mathcal{H}=\Delta\mathcal{H}_1+\Delta\mathcal{H}_2$, where

$$\Delta\mathcal{H}_1(\Phi)=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2\Omega^2+\Gamma^2}\int_x[F_n^2(\Phi(x))+G_n^2(\Phi(x))],$$

$$\Delta\mathcal{H}_2(\Phi)=\sum_{n=1}^{\infty}\frac{\Gamma}{(n^2\Omega^2+\Gamma^2)(2n\Omega)}\int_x\mathcal{W}[F_n(\Phi(x)),G_n(\Phi(x))]. \tag{21}$$

$\mathcal{W}[F_n(\Phi(x)),G_n(\Phi(x))]$ is defined by the indefinite integral $\mathcal{W}[F(\alpha),G(\alpha)]=\int d\alpha[G(\alpha)F'(\alpha)-F(\alpha)G'(\alpha)]$. After integrating out Π in Eq. (18), we finally get

$$\overline{\lim_{t\to\infty}\langle\varphi(x_1,t)\varphi(x_2,t)\cdots\rangle_\eta}=\frac{1}{Z}\int\mathcal{D}[\varphi(x)]\varphi(x_1)\varphi(x_2)\cdots e^{-\mathcal{H}_{\rm eff}(\varphi)}, \tag{22}$$

where $Z=\int\mathcal{D}\varphi\exp(-\mathcal{H}_{\rm eff})$ and $\mathcal{H}_{\rm eff}(\varphi)=\mathcal{H}(\varphi)+\Delta\mathcal{H}(\varphi)$. The effective energy functional for a N-component order parameter field $\varphi\equiv\{\varphi_a\}$ is a straightforward generalization of Eqs. (20) and (21). In this case $\Delta\mathcal{H}$, which satisfies Eq. (20) for each component, exists only if the components of the driving field F_n^a and G_n^a satisfy the conditions $\partial F_n^a/\partial\varphi_b=\partial F_n^b/\partial\varphi_a$ and $\partial G_n^a/\partial\varphi_b=\partial G_n^b/\partial\varphi_a$.

Note that we have neglected $O(\xi^2)$ terms in the expansion of the $\mathcal{L}$ and $\Delta\mathcal{L}$ operators because ξ is assumed to be small. The condition for the smallness of ξ and the criteria for the validity of $O(\xi)$ approximation are both obtained by comparing the neglected terms with those that are retained in these operators. For instance, if $\mathcal{F}=2\tilde{g}\varphi^2(x)\cos(\Omega t)$ and the coefficient of φ^4 term in $\mathcal{H}(\varphi)$ is λ then ξ is small if $\lambda \gg \tilde{g}^2/(\Omega^2+\Gamma^2)$. A self-consistent criteria can also be obtained by comparing the expectation values of the neglected and the retained terms using $\mathcal{H}_{\text{eff}}$. In some cases the higher powers of ξ contribute only irrelevant terms to $\mathcal{H}_{\text{eff}}$ and hence can be neglected while determining universal properties, independent of ξ being small. For instance, if $\mathcal{F}=\tilde{g}\varphi^4(x)\cos(\Omega t)$ and $\mathcal{H}$ is a φ^4 theory in four dimensions, then the $O(\xi)$ term in $\mathcal{H}_{\text{eff}}$ itself is an irrelevant φ^8 field and the higher-order terms contain fields that are more irrelevant.

The effective energy functional $\mathcal{H}_{\text{eff}}$ can be interpreted as follows. The asymptotic distribution contains both statistical fluctuations Φ and the dynamical fluctuations $\xi(\Phi,t)$ that are periodic in time. In Eq. (20), if we substitute $\mathcal{F}$ in terms of ξ by using Eq. (11) and then integrate by parts, we get

$$\frac{\delta}{\delta\Phi}\Delta\mathcal{H}(\Phi) = \frac{\partial}{\partial\Phi}\frac{1}{2}\overline{(\partial_t\xi)^2} + \Gamma\overline{\partial_t\xi\frac{\partial}{\partial\Phi}\xi}. \qquad (23)$$

Thus, by averaging over the time period we have eliminated the dynamical fluctuations and provided an effective description to the system in terms of the statistical fluctuations that are governed by a modified energy functional $\mathcal{H}_{\text{eff}}=\mathcal{H}+\Delta\mathcal{H}$.

IV. ILLUSTRATIVE EXAMPLES

We have seen that the large-time behavior of periodically driven stochastic systems, averaged over a period, can be described by an effective LG functional of equilibrium systems. We can now make use of the knowledge about these equilibrium systems to induce new phases and phase transitions in various systems by subjecting them to time-dependent periodic fields. We illustrate some of the effects due to these driving fields with a few examples.

A. Varying the critical point

The driving fields transform the energy functional $\mathcal{H}$ into a new energy functional $\mathcal{H}_{\text{eff}}$. This amounts to changing the coupling constants $\{g\}\rightarrow\{g_e\}$ which in turn induces a change in the critical point. We will illustrate this with an example. The LG functional that describes the Ising model near the critical temperature is

$$\mathcal{H}[\varphi] = \int_x (\partial\varphi(x))^2 + A(T-T_c)\varphi^2(x) + \lambda\varphi^4(x). \qquad (24)$$

The mean-field theory suggests a second order transition at T_c from a Z_2-symmetry-broken phase to the symmetry-unbroken phase and a dependence of magnetization (order parameter) on temperature below T_c as $\langle\varphi\rangle^2=A(T_c-T)/2\lambda$. If we now drive this system, say, by oscillating φ^2 term, which is same as adding a force term

$$\mathcal{F}(\varphi(x)) = 2\tilde{a}\varphi(x)\cos(\Omega t), \qquad (25)$$

then this system will get described by the following effective energy functional that is obtained using Eq. (21)

$$\mathcal{H}_{\text{eff}}[\varphi] = \int_x (\partial\varphi(x))^2 + a_e\varphi^2(x) + \lambda\varphi^4(x), \qquad (26)$$

where $a_e=A(T-T_c)+\tilde{a}^2/(\Omega^2+\Gamma^2)$. Since this effective energy functional differs from the original functional only in the coefficient of the φ^2 term one can read off the critical temperature θ_c and the behavior of the time-averaged magnetization $\overline{\langle\varphi\rangle}$ below θ_c of the driven system. We get

$$\theta_c = T_c - \frac{\tilde{a}^2}{A(\Omega^2+\Gamma^2)},$$

$$\overline{\langle\varphi\rangle}^2 = \frac{A}{2\lambda}(\theta_c - T). \qquad (27)$$

Hence, the driving field acting on this system tends to destroy the symmetric phase as it reduces both the critical temperature and the magnetization at a given temperature.

B. Changing the nature of transition

The nature of phase transition can also be changed by applying driving fields. Consider the LG functional as given in the previous example, Eq. (24), but with a different driving force

$$\mathcal{F}(\varphi(x)) = 2(\bar{a}\varphi(x) - \bar{b}\varphi^3(x))\cos(\Omega t), \qquad (28)$$

and also assume the dimension of space $d>3$ where φ^6 term is irrelevant. The effective energy functional will then become

$$\mathcal{H}_{\text{eff}}[\varphi] = \int_x (\partial\varphi(x))^2 + a_e\varphi^2(x) + \lambda_e\varphi^4(x) + b_e\varphi^6(x), \qquad (29)$$

with the coupling constants $a_e=A(T-T_c)+\tilde{a}^2/(\Omega^2+\Gamma^2)$, $\lambda_e=\lambda-2\tilde{a}\tilde{b}/(\Omega^2+\Gamma^2)$, and $b_e=\tilde{b}^2/(\Omega^2+\Gamma^2)$.

Now the mean-field prediction is as follows. For $\lambda_e>0$ two different phases exist; a spontaneously broken phase when a_e is negative and an unbroken phase when a_e is positive. These phases are separated by a line of second order transition which ends in a tricritical point when both λ_e and a_e become zero. For $\lambda_e<0$ there are three possibilities depending on the value of a_e. (i) If $a_e<\lambda_e^2/4b_e$ then it is in a symmetry broken phase. (ii) If $\lambda_e^2/4b_e<a_e<\lambda_e^2/3b_e$ then it is a metastable phase and will have a broken symmetry if one enters this region crossing the line $\lambda_e^2=4a_eb_e$ and will have an unbroken symmetry if one enters by crossing the line $\lambda_e^2=3a_eb_e$. (iii) If $a_e>\lambda_e^2/3b_e$ then it is in the symmetric phase. The line $\lambda_e^2=4a_eb_e$ is a line of first-order transition which ends in the tricritical point.

Suppose the system initially is in the symmetric phase with both a_e and λ_e positive. Now switch on the periodic force such that the product $\tilde{a}\tilde{b}$ is positive and then gradually reduce the driving frequency Ω. This will reduce both a_e and λ_e and takes the system across the line of first-order transition into the symmetry broken phase. More generally, by tuning at most two of the three parameters $\{\tilde{a},\tilde{b},\Omega\}$ and the temperature T we can scan the entire $a_e-\lambda_e$ plane. Hence, by applying the driving fields we can engineer the behavior of the system.

C. Inducing new fixed points

Driving fields can enlarge the coupling constant space of the system by introducing either relevant or irrelevant fields. Though the irrelevant fields usually do not change the large-distance properties of the system but in some cases, such as in the previous example, the coupling constant of the relevant fields can get shifted to a region where irrelevant fields become important. Let us now examine an example where the driving fields introduce a relevant field and drastically alter the large-distance properties. This is because the system flows under scaling to a new stable fixed point in the enlarged coupling constant space. The system under study is described by the $O(N)$ model

$$\mathcal{H} = \int_x (\partial\varphi(x))^2 + t\varphi^2(x) + u(\varphi^2(x))^2, \tag{30}$$

where $\varphi \equiv \{\varphi_1,\ldots,\varphi_N\}$ is a N-component vector field. This model has two fixed points: one is at $u=0$ (Gaussian fixed point) and the other is at finite u (Heisenberg fixed point). For dimension $d<4$ Heisenberg fixed point is stable while Gaussian is not. Now drive the system by the fields

$$\mathcal{F}_a(\varphi(x)) = 2\varphi_a^2(x)\cos(\Omega t), \tag{31}$$

$a=1,\ldots,N$. The resultant model is the $O(N)$ model with a cubic symmetry breaking term [12]

$$\mathcal{H}_{\text{eff}} = \mathcal{H} + v\int_x \sum_a \varphi_a^4(x), \tag{32}$$

where $v=(\Omega^2+\Gamma^2)^{-1}$. In the $u-v$ plane this model has two more fixed points, Ising fixed point at $(u=0,v\neq 0)$ and the cubic fixed point at $(u\neq 0,v\neq 0)$, apart from the Gaussian and Heisenberg fixed points at $(u=0,v=0)$ and $(u\neq 0,v=0)$, respectively. The stability of the fixed points is as follows: The Gaussian point is unstable in the v direction too. Ising point is stable in the v direction but unstable in the u direction. Heisenberg point is stable in the v direction if $N<4$ and unstable if $N>4$. Cubic point is stable in both directions if Heisenberg point is unstable and vice versa. Thus, when $N>4$ these driving fields change the large-distance properties of the system from Heisenberg to cubic.

Periodically driven stochastic models with $O(N)$ symmetry, without the $m\partial_t^2\varphi(x,t)$ in Eq. (2), were first studied in the context of magnetic hysteresis [13,14]. The phase transitions that get induced by the driving fields in these models were recently investigated too [15]. These transitions were also observed recently in Monte Carlo simulations of a driven kinetic Ising model [16] and were subsequently analyzed within a periodically driven TDLG model [17].

In summary, we have derived the effective theory for the correlation functions of a class of periodically driven macroscopic systems. We have shown with a few examples that this effective theory can be made use to select the driving fields that can steer the system through a plethora of phases.

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